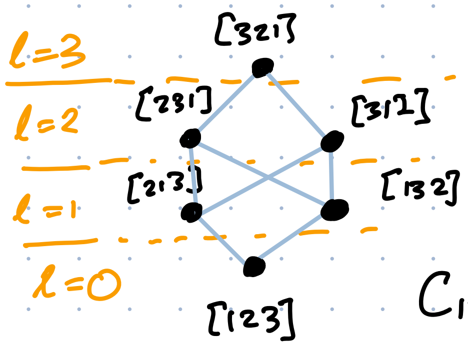
Lecture 17

Example $G = SL_3\mathbb{C}$ $\quad W = Sym_3 =$

[123]	[213]	[132]	[231]	[312]	[321]
e	a	b	ba	ab	$aba = bab$
	s_α	s_β			w_0



$C_{123} = \{x_0\}$ $\qquad C_{213} \cong C_{132} \cong \mathbb{C}$ $\qquad X_{213} \cong X_{132} \cong \mathbb{CP}^1.$

$G/B = \text{flags in } \mathbb{C}^3 = \{(p,\ell) \mid p \text{ point in } \mathbb{CP}^2,\ \ell \text{ line in } \mathbb{CP}^2\}$

$(v_1 | v_2 | v_3)\cdot B \leftrightarrow [v_1] \in \mathbb{CP}^2,\ \overline{[v_1][v_2]} \subset \mathbb{CP}^2 = [\text{span } v_1, v_2]$

$$C_{213} = B\cdot\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\cdot B = \begin{pmatrix}1&*&*\\0&1&*\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}B = \begin{pmatrix}*&1&0\\1&0&0\\0&0&1\end{pmatrix}B$$

"upward" row ops

$$= \{(p,\ell) \mid \ell = [\text{span } e_1, e_2],\ p \neq [e_1]\}$$

$$X_{213} = \{(p,\ell) \mid \ell = \text{span}[e_1,e_2]\} \cong \mathbb{CP}^1 = \mathbb{P}(\text{span}(e_1,e_2))$$

$$C_{132} = B\cdot\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}B = \begin{pmatrix}1&*&*\\0&1&*\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}B = \begin{pmatrix}1&*&*\\0&*&1\\0&1&0\end{pmatrix}B = \begin{pmatrix}1&0&0\\0&*&1\\0&1&0\end{pmatrix}B$$

result of upward row ops $\qquad$ result of rightward col ops.

$$= \{(p,\ell) \mid p = [e_1],\ \ell \neq [e_1,e_2]\}$$

$$X_{132} = \{(p,\ell) \mid p = [e_1]\}.$$

Similar analysis: $X_{231} = \text{closure of } \begin{pmatrix}*&1&0\\*&0&1\\1&0&0\end{pmatrix}B = \{[e_1] \in \ell\}$

$X_{312} = \{p \in [e_1,e_2]\}$

$C_{321} = \text{span } v_1 e_2 e_3 = \text{span } v_1 v_2 e_3 = \mathbb{C}^3 \qquad X_{321} = Flag(\mathbb{C}^3).$

Note

$$\begin{array}{ccc} & \mathrm{Flag}(\mathbb{C}^3) = G/B & \\ \swarrow & & \searrow \\ G/P_\alpha & & G/P_\beta \\ (\mathbb{CP}^2)^* & & \mathbb{CP}^2 \end{array}$$

And both are $\mathbb{CP}^1$ bundles. The 1-dim Schubert vars are fibers of these.

So: $\mathrm{Flag}(\mathbb{C}^3) = \mathbb{CP}^1 \wedge \mathbb{CP}^1 \cup \mathbb{C}^2 \cup \mathbb{C}^2 \cup \mathbb{C}^3$

Exercise. Attaching map desc? $S^3 \longrightarrow \mathbb{CP}^1 \wedge \mathbb{CP}^1$?

$$H_k(\mathrm{Flag}(\mathbb{C}^3)) = \begin{cases} \mathbb{Z} & k = 0, 6 \\ \mathbb{Z}^2 & k = 2, 4 \\ 0 & \text{else} \end{cases}$$

Example $G = SL_n\mathbb{C}$. $W = \mathrm{Sym}_n$. Gen by $\alpha_1 \dots \alpha_{n-1}$

$\Phi^+ = \{e_i - e_j \mid i < j\}$ α_i = switch $i, i+1$.

$w_0 = (n \;\; n-1 \;\; n-2 \;\; \dots \;\; 1)$ $\ell(w_0) = |\Phi^+| = \frac{n(n+1)}{2} = \dim_{\mathbb{C}} G/B = \dim_{\mathbb{C}} \mathrm{Flag}\,\mathbb{C}^n$.

$\sigma \in W$ view as map $[n] \to [n]$.

. . . . . .

The left action of B on a matrix doesn't change the dim of $F_i \cap E_j$ where F_i = span cols $1 \dots i$ E_j = span $e_1 \dots e_j$

Let $d_{ij} = \dim(\sigma E_i) \wedge E_j$ = # 1 entries in the $j \times i$ upper left submatrix of σ

$= \#\{k \mid \sigma(k) \leq j,\ k \leq i\}$

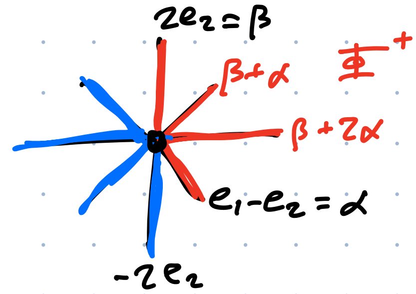
Then $X_{ij} = \{F \mid \dim F_i \cap E_j \geq d_{ij}\}$ C_{ij} = equality not $\geq$

e.g. $d_{ij} = n - i - j$ is "expected" and corresp to w_0, the open cell

$d_{ij} = \min(i,j)$ is when $F_i = E_i$. $\sigma = e$ point!

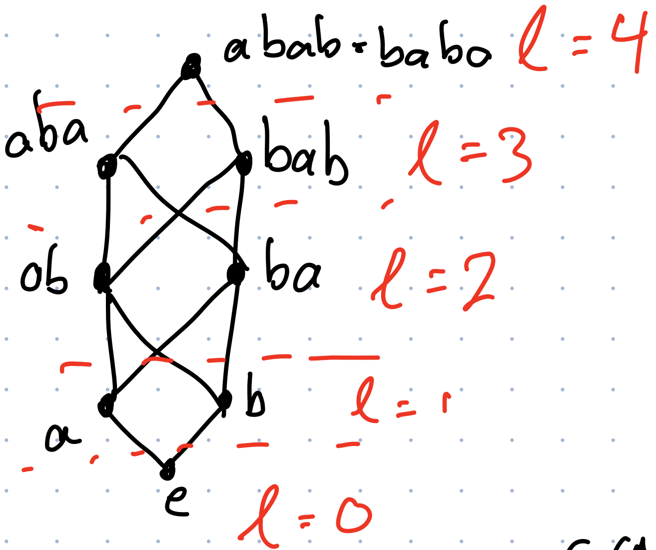
Example. (see lec 10) $G = Sp(4,\mathbb{C})$ $\quad W \cong (\mathbb{Z}/2)^3$

G/B = isotropic flags



$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ $\quad dz_1 \wedge dw_1 + dz_2 \wedge dw_2$

(z_1, z_2, w_1, w_2)



$a = s_\alpha \quad b = s_\beta \quad a(\alpha) = -\alpha$

$a(\beta) = \beta + 2\alpha$

$b(\alpha) = \alpha + \beta \quad b(\beta) = -\beta$

$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}_a \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}_b$ Using Δ as basis

$\sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} ab$

$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \right\}$ B, C sym

$e_i(t) = t_i \quad e_1, e_2$ basis $\mathfrak{h}^*$

$Sp(4,\mathbb{C})$. $\mathfrak{h} = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & -t_1 & \\ & & & -t_2 \end{pmatrix} \middle| \; t_1, t_2 \in \mathbb{C} \right\}$. rank = 2.

$\dim \mathfrak{g} = 10$ Basis for compl of $\mathfrak{h}$:

$\{ a_{12} = 1,\ a_{21} = 1,\ b_{11} = 1,\ b_{22} = 1,\ c_{11} = 1,\ c_{22} = 1,\ c_{12} = c_{21} = 1,\ b_{12} = b_{21} = 1 \}$

Roots: $e_1 - e_2 \quad e_2 - e_1 \quad 2e_1 \quad 2e_2 \quad -2e_1 \quad -2e_2 \quad -e_1 - e_2 \quad e_1 + e_2$

To find cells, need the matrices representing s_α, s_β as elements of $N(H)/H$

s_α corresp to matrix M_a s.t. $M_a \mathfrak{g}_\alpha M_a^{-1} < \mathfrak{g}_{-\alpha}$ $\quad M_a H M_a^{-1} = H$

$M_a \mathfrak{g}_\beta M_a^{-1} < \mathfrak{g}_{\beta + 2\alpha}$

$\mathfrak{g}_\alpha = \begin{pmatrix} 0 & t & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & -t & 0 \end{pmatrix} \quad \mathfrak{g}_{-\alpha} = \begin{pmatrix} 0 & 0 & & \\ t & 0 & & \\ & & 0 & -t \\ & & 0 & 0 \end{pmatrix} \quad \mathfrak{g}_\beta = \begin{pmatrix} & & 0 & 0 \\ & & 0 & 1 \\ & & & \\ & & & \end{pmatrix} \quad \mathfrak{g}_{\beta+2\alpha} = \begin{pmatrix} \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & 0 & 0 \\ & & & \\ & & & \end{pmatrix}$

Conclude: $\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} = M_a$

M_b s.t. $M_b \mathfrak{g}_\alpha M_b^{-1} \subset \mathfrak{g}_{\alpha+\beta}$

$\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & -1 & 0 \end{pmatrix} \to \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ & & & \\ & & & \end{pmatrix} \qquad \begin{pmatrix} & & 0 & 0 \\ & & 0 & 1 \\ & & & \\ & & & \end{pmatrix} \to \begin{pmatrix} & & & \\ & & & \\ 0 & 0 & & \\ 0 & 1 & & \end{pmatrix} \cdot$ const

$e_1 - e_2 = \alpha$ $\qquad$ $\alpha+\beta$, $e_1 + e_2$

An example of such M_b is $\begin{pmatrix} 1 & & & \\ & & & -1 \\ & & 1 & \\ & 1 & & \end{pmatrix}$ note $M_b^2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \in H$

but $M_b^2 \neq I$.

$B M_a B = \begin{pmatrix} 1 & * & & * \\ & 1 & & \\ & & 1 & \\ & & * & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & * & & * \\ 0 & 1 & & \\ & & 1 & 0 \\ & & * & 1 \end{pmatrix}$

$= \begin{pmatrix} * & 1 & * & * \\ 1 & 0 & * & * \\ 0 & 0 & ? & \\ 0 & 0 & & \end{pmatrix} = z_1$ axis is in F_2 & $F_1 \ni z_1$ axis

$B M_b B = ?$ Exercise.

$\begin{pmatrix} 1 & * & & * \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -* & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 1 & * & \cdot & \cdot \\ 0 & * & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot \end{pmatrix} B = F_1$